**Units of Measure**

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**Introduction**

The broader the circle of colleagues you have, the more important precision in writing and speech becomes. It is quite common for people to speak of measurements without including the underlying units of measurement. That’s because we make assumptions that others know what we mean. When we say that a child is “four six and sixty eight”, we can deduce these must be measured in feet, inches, and pounds, for these figures do not make much sense for a child using any of the others systems that we use.

As we move away from situations where one can easily deduce the units of measure from context and the people we work with use other measurement systems, being explicit about the units becomes important, if not critical. The Mars Climate Orbiter spacecraft was lost due to a failure to resolve a metric and English incompatibility.[[1]](#footnote-1)

**Basis for measurement**

Measurement of physical properties is purely cultural. Various groups assert standards and then it is up to the rest of us to decide which of these standards we will honor. While everyone seems to agree that it would be helpful if we could agree on a single standard, and many have tried, cultural norms are very difficult to change. Therefore, it is even more important to be specific about the units of measure when one deals cross culturally.

The kinds of things that scientists measure tend to be related, and therefore it is common for a set of *base dimensions* to be defined upon which a set of *derived dimensions* can be built[[2]](#footnote-2). It is common for mass (M), length (L), and time (T) to serve as the base dimensions and for other dimensions to be built up from these, such as in the case of *force*, which can be derived from a set of base measures, M ∙ L ∙ T-2. (The raised dot, “∙”, signals multiplication when its absence might be ambiguous. The “T -2” signals a reciprocal value of time as in “per second squared”.)

Therefore, as we compute and interact with others, we must select the base dimensions that we will use and then which units of measure will be employed for measurements of those dimensions. Failure to do this carefully can easily lead to a needless loss.

**The Metric System**

The Metric System (abbreviated as SI from the French title *Le Système International d’Unités*) is the closest thing we have to an international standard for units of measure. The United States government has documented this standard in the form of a guide for its use and it is a good reference for authors.[[3]](#footnote-3) This guide is a valuable tool for scientists and those working with them.

Page four of this guide lists the following seven standard base dimensions in the SI, from which all other common dimensions are derived.

|  |  |  |
| --- | --- | --- |
| Base quantity | Name | Symbol |
| 1. length | meter | m |
| 2. mass | kilogram | kg |
| 3. time | second | s |
| 4. electric current | ampere | A |
| 5. thermodynamic temperature | kelvin | K |
| 6. amount of substance | mole | mol |
| 7. luminous intensity | candela | cd |

Table 1: SI base units

The following table lists derived units. For each of these quantities, you can see how the units are built up using the base units.

|  |  |  |
| --- | --- | --- |
| Derived quantity | Name | Symbol |
| 1. area | square meter | m2 |
| 2. volume | cubic meter | m3 |
| 3. speed, velocity | meters per second | m/s |
| 4. acceleration | meters per second squared | m/s2 |
| 5. wavenumber | reciprocal meter | m-1 |
| 6. density, mass density | kilogram per cubic meter | kg/m3 |
| 7. specific volume | cubic meter per kilogram | m3kg |
| 8. current density | ampere per square meter | A/m2 |
| 9. magnetic field strength | ampere per meter | A/m |
| 10. luminance | candela per square meter | cd/m2 |
| 11. amount-of-substance concentration,  amount concentration, concentration | mole per cubic meter | mol/m3 |

Table 2: SI derived units

The metric system does not, however, specify how to measure **all** things that will be of interest to scientists, business people, or the average people of our world.

**Measurements beyond the metric system**

It is easy to be overwhelmed by the size of the NIST document on the metric system, and to think that every possible unit of measurement has been captured there. But this is not true. For example, let’s imagine that you are building a report for a client on the market for replacement toilets in homes. Knowing the number of toilets per home and the number of homes per square kilometer would be most valuable information. What is the standard name for these units? The word “toilet” does not appear in this 90-page document. Neither do the words “home”, “house”, or “dwelling”. This does not suggest that units to represent these concepts are not needed.

The Metric System and the NIST guide give us a foundation upon which to build more complex units that we might need for our application. There is an impossibly long list of things for us to count, and each time we write down such counts, it is important to remind ourselves and those who might read our work, what the number represents. If there are 318 homes in a certain square kilometer, the base of the counting (the number of homes) should be recorded. It is also desirable to record that this is not the total number of homes. Rather, it is the number of homes within a square kilometer.

318 homes (1)

Let’s start by recording the count by creating the unit “homes” and labeling the number 318 with this label as in formula (1). This represents a counting of homes and just a label such as this could be used for anything that we might count.

In this case, there is actually more information here than just the fact that there are 318 homes. We also know that this is within a certain square kilometer. The number of something in a certain area is typically referred to as a “density”. The density of homes in this particular area is 318 homes per square kilometer and it would typically be written in one of two ways.

318 homes / km2 (2)

318 homes ∙ km-2 (3)

Both formula (2) and (3) express the same thing, but it is probably more common to use the (2) form over the (3) form. Densities are being expressed when you see a unit in one of the following two forms, where “L” is a unit of some length:

<count> / L2 (4)

<count> ∙ L-2 (5)

Continuing on this example, we have also counted the number of toilets in these homes by going to each and every home and asking the person who answered the door how many toilets are in this home. The total number of toilets in this particular square-kilometer was 507 and therefore, it would be wise to properly label the value 507 with the word “toilets”.

507 toilets (6)

Similarly, since we know that the number of toilets is from an area of a known size, we can specify the density of toilets in this area. So we could write that fact out in one of the following two ways:

507 toilets / km2 (7)

507 toilets ∙ km-2 (8)

If someone wanted to know the average number of toilets per house, we would divide the total number of toilets by the total number of homes. For our case, we should record this computation by notations similar to the following:

507 toilets ÷ 318 homes (9)

1.594339622641509 toilets / home (10)

Now we need to address the issue of how many significant digits to the right of the decimal point should be recorded. It is true, to the best of our knowledge, that both the 507 and the 318 are exact values; so dividing 507 by 318 does, indeed, result in an exact result. In this case, my calculator provides me with a sequence of 16 digits (if we choose to believe it). For the purpose of observing neighborhood trends, recording any more than one or two decimal places is probably not helpful. In this case, it probably makes sense to round the result to just one decimal place, as we cannot imagine how any more of those digits would change or bring value to us moving forward. Dragging around all of those digits takes time and effort and this suggests that maybe there is some reason for keeping them all, which is not true.

A second issue is the change from “homes” to “home” in the units for formula (10).

The NIST guide is very clear that we **do not** use plurals for the standard units **symbols**[[4]](#footnote-4), so that we would **not** record fifteen kilometers as “15 kms”. (It should be written as “15 km”.) The guide goes on, however, to tell us that when writing out these names in full, we use the normal English rules for plurals[[5]](#footnote-5). Since “homes” and “home” are written-out English words and not symbols, the normal rules for plurals apply.

When one writes about the average number of items of A with respect to B, we typically write the units as “A / B” and write it out as “A per B” in and “1.6 toilets per home” in this case, and the B should be singular.

If we wanted to express this average as a density figure, in order to compare it with other densities, we would write it as follows.

1.6 toilets / home / km2 (11)

In this case, the meaning is left associative. In other words, we start on the left side and meaning is established working left to right. The number 1.6 refers to toilets, so the word toilet must come next. Now 1.6 in this context is an average number of toilets, so we express that average by adding “/ home” to the units. Then, to express that this average is a density, we must add the units to express the area over which the average applies and we would add that to the end of the units as in “/ km2”. To make this explicit[[6]](#footnote-6), one could use parentheses as in “((1.6 toilets) / home) / km2”.

There is meaning in the ordering of the units. While “1.6 km-2 ∙ home-1 ∙ toilet” might be mathematically correct, it does not properly convey the meaning “1.6 toilets / home / km2” does.

**Addition and Subtract with Units**

A good friend, William Riddle, captured one of my favorite visual jokes[[7]](#footnote-7) in the state of Colorado a number of years ago, and I have used it many times in my teaching. Having a set of numbers does not suggest that their sum has any meaning and this photo makes this point in a delightful way.

Macintosh HD:Users:LRCarter:Desktop:Gold Hill 8x10.pdf

Figure 1: Invalid addition of values

When we wish to add or subtract values, they must be compatible. In the case of the Gold Hill sign, the values are not compatible and so it makes no sense to add these numbers together and report the sum.

One way that things can be compatible is when they have exactly the same units. Give a set of small boxes of different dimensions; one can compute the total volume of the boxes, by computing the volume of each box, using the unit label cm3. It makes perfect sense, then, to compute the total volume by adding each of these volume values, and label the result with the sum.

Just because things are addition or subtraction compatible does not mean that addition or subtract makes sense. In some cases, we are called upon to compute intermediate results to produce something that does make sense. For example, assume we have five samples from a city and have computed the average number of toilets per house for each of these five locations. Each of these averages has a unit specification of “toilets / house”. What meaningful result is computed by adding these five things together? The sum, by itself, really means nothing. As a part of a longer process, however, adding these averages together and then dividing by five, we now have a new average of averages, and that does make sense.

There are two cases where values are compatible for addition and subtraction when the units are not identical. The first is where the units are composed of base units by means of multiplication, and the ordering of the units are not the same.

For example, *force* is expressed as “M ∙ L ∙ T-2”. What happens if we have force expressed the usual way in one value and another specified as “L ∙ M ∙ T-2”? Are these compatible? The answer is yes, even though the meaning may not be as clear. We humans see that the order of the computation is different from what we would normally do, and adjust the units to one of our standard derived forms. In this case we are used to force as being computed as mass times acceleration, which leads us to “M ∙ L ∙ T-2” from “m ∙ a”, where “m” is “mass” and “a” is acceleration (“L ∙ T-2”).

The second instance where values are compatible for additional and subtraction, when the units are not identical, is when one of the units of measurement are based on a different standard. For example, it is possible to compute the average speed of an object, even when some of the distances are measured in feet and others are measured in miles. In this situation, the values will have to be normalized to a common set of base measures before the addition or subtraction can be performed.

For example adding velocities is possible, as long as the units of the base measures are the same. If a gun is fired forward from a speeding car, we can compute the actual initial velocity of the bullet by adding the velocity of the car to the muzzle velocity of the bullet. If a modern rifle fires a bullet at 1,000 m/s and the car is traveling in the same direction at 200 k/h, what is the actual muzzle velocity from the perspective of a stationary observer? It is not proper to add 1,000 to 200 in this case, since the units do not match. We can, however, convert one of these velocity figures to the units of the other. We use the word *normalizing* to describe this process.

In this case, since we are working with the speed of the bullet and they are typically expressed in units of “m/s”, we need to convert the car’s velocity into meters per second. A kilometer is 1,000 meters and there are 3,600 second per hour (60 second/minute times 60 minutes / hour). So the following sequence performs the required computation.

200 k/h × 1,000 m/k = 200,000 m/h Convert k/h to m/h (12)

200,000 m/h ÷ 3,600 s/h = 56 m/s Convert m/h to m/s (13)

1,000 m/s + 56 m/s = 1,056 m/s Add the normalized velocities (14)

When we use units properly, having mixed units from differing base standards does not cause problems, as long as we honor the units. In many ways, the process of adding 57 lb to 20 kg is similar to adding ½ to ⅓. Before addition can be performed, the values must be normalized. We have to convert pounds to kilograms or kilograms to pounds. (In the case of ½ and ⅓, normalization results in expressing these values ³⁄₆ and ²⁄₆ and now they can be added together producing ⁵⁄₆ as the result.)

**Multiplying with Units**

When the algorithm calls for multiplication of values with units, things are not the same as for addition. Given a value in units “A” and another value in units “B”, it is possible to multiply the two values producing a product in units “A ⋅ B”. In the case where both values are based in units of “A”, the product will be “A ⋅ A” which is usually written as “A2”. Consider multiplying two lengths: 2.3 meters and 1.6 meters. We expect the result to be an area with a unit specification of m2.

2.3 m A first value (15)

1.6 m A second value (16)

2.3 m× 1.6 m = 3.68 m2 A product that is an area (17)

Now, consider multiplying this area by another length, 3.2 meters. We expect the result to be a volume with a unit specification of m3.

3.68 m2 A first value (18)

3.2 m A second value (19)

3.68 m2 × 3.2 m = 11.776 m3 A product that is a volume (20)

We have learned in the previous section that some units, composed of a set of factors, are typically expressed with the factors in a certain specified order. For example, *force* is usually expressed as “M ∙ L ∙ T-2” and some readers might be confused if it were written as “L ∙ M ∙ T-2”, even thought mathematically, they are the same. One way of detecting a beginner is by their failure to recognize that an expression of units should be reordered into one of the canonical[[8]](#footnote-8) forms.

When multiplying factors, it is possible for units to cancel. Consider taking an acceleration, expressed with units “L ∙ T-2”, and multiplying it by a time value with units “T”. The product will be a velocity at that instant with units “L ∙ T” as in the following.

9.80665 m/s2 Gravity acceleration at sea level (21)

5 s A duration (22)

9.80665 m/s2 × 5 s = 49.03325 m/s A product that is a velocity (23)

It is common to say that the seconds in the duration *cancels* one of the seconds in the acceleration producing in a velocity. We saw this same kind of cancellation in the case of (12) above where the kilometer in the unit’s numerator on the left value canceled the kilometer in the unit’s denominator in the right value, producing a product unit of “m/h”.

Now consider the situation where we take a given fixed velocity (not an acceleration) and multiply it by a fixed duration. We expect the result to be a length with a unit specification of m.

49 m/s A fixed velocity (24)

2.5 s A duration (25)

49 m/s× 2.5 s = 123 m A product that is a length (26)

Once again, the seconds cancel, leaving just a length.

**Division with Units**

A common definition of ‘A divided by B’ is, how many times B is in A. The division algorithm we learned in school is actually a form of repetitive subtraction. With this definition, we would assume that the units of the divisor, B, must be the units of the dividend, A, or a factor of the units in the dividend. An example of this kind of problem is found when trying to determine how many small boxes of rice can be filled from a large storage bin of rice[[9]](#footnote-9). If each small box can hold at most 1240 cm3 of rice and the storage bin holds 30 × 106 cm3 of rice, the minimum number of boxes required would be computed as follows.

30 × 106 cm3/bin Storage bin volume (27)

1,240 cm3/box Box volume (28)

30 × 106 cm3/bin ÷ 1,240 cm3/box = 24,193.55 box/bin (29)

The calculation of the number of boxes per bin employs division, but it is easier to see how to produce the units by inverting the denominator and multiplying, as shown below.

30 × 106 cm3/bin ÷ 1,240 cm3/box (30)

30 × 106 cm3/bin × 1/1,240 box/cm3 (31)

(30 × 106/1,240) (cm3/bin × box/cm3) (32)

24,193.55 (cm3/bin × box/cm3) (33)

24,193.55 box/bin (34)

We know from math that *x* ÷ *y* is the same as *x* × 1/*y*. We use this rule to transform formula (30) into formula (31). Notice what happens to the units. The unit specification was “cm3/box” before we did the inversion, so the resulting unit specification of “box/cm3” makes since, as it is “upside down” (the old numerator is now the denominator and the old denominator is now the numerator).

Computing the result requires us to multiply the values and “multiply” the units. We see the preparation for that step by the reordering and grouping with parentheses that was done in formula (32). The computation of the numeric portion of the value can be done with a simple calculator and the results of that, rounded to two decimal places, is shown in formula (33). The “multiplication” of the units, where the “cm3” terms cancel, is shown in formula (34) with the same result as that in formula (29).

The notion that division is about finding how many of one item is in another is not helpful for many scientific operations. The concepts of *velocity* and *acceleration* employ division where there is no relationship between the units in the numerator and the denominator. We define velocity in terms of L / T. For example, if it takes 2.5 hours to travel 225 kilometers, we can compute the average velocity by dividing the distance traveled (the L) by the time it took to travel that distance (the T).

225 km / 2.5 hours (35)

(225/2.5) (km/hours) (36)

90 km/h Commonly written 90 kph (37)

In those situations where there are no units in common the division of two values results in units expressed with a “/” in the units and the word “per” used when it is written out. This is clearly seen with acceleration. Acceleration is change in velocity per unit of time, just as velocity is a change of location (a length) per unit of time. L / T2 is the standard form for expressing acceleration, but one can think of it as (L/T)/T.

To compute an average acceleration, compute the difference of the velocity at points A and B, and then divide that difference by the time it took to go from point A to point B. For example, consider a vehicle that is traveling 90 km/h at point A, is traveling 140 km/h at point B, and it took 10 seconds to go from point A to point B, the acceleration would be computed as follows.

(140 km/h – 90 km/h) / 10 s The starting equation (38)

50 km/h / 10 s Compute the difference in velocities (39)

(50 km/h / 10 s) × (1,000 m/km) Normalize kilometers to meters (40)

(50 km/h) × (1,000 m/km) / 10 s Associate factors (41)

(50 × 1,000) (km/h × m/km) / 10 s Associate factors (42)

(50,000 m/h) / (10 s) Convert km/h to m/h (43)

(50,000 m/h) / (10 s) × (1/3,600 h/s) Normalize hours to seconds (44)

(50,000 m/h) × (1/3,600 h/s) / (10 s) Associate factors (45)

((50,000 × 1/3,600) (m/h × h/s)) / (10 s) Distribute the factors (46)

((50,000/3,600) (m/h × h/s)) / (10 s) Compute the product (47)

(13.89 (m/h × h/s)) / (10 s) Compute the quotient (48)

(13.89 m/s) / (10 s) Compute the units (49)

(13.89/10) (m/s/s) Compute the acceleration (50)

(1.389) (m/s/s) Compute the quotient (51)

1.389 m/s2 Compute the units (52)

We have seen that division is truly just multiplication by the inverse in both the value and the units. Unlike addition and subtraction, multiplication and division can be performed with any units, producing new units. While the results might be “valid” they may not be meaningful, since new units may or may not be meaningful. The significance of the units depends upon the algorithm being employed.

**Conclusion**

There are many more mathematical operations than what has been covered here. We believe that this introduction should be adequate to provide you with a solid foundation to understand what others have done, or enable you to extend these operations to whatever you require.

1. http://mars.jpl.nasa.gov/msp98/news/mco990930.html [↑](#footnote-ref-1)
2. Kennedy, Andrew J., “Relational Parametricity and Units of Measure”, in the *Proceedings of the 24th Annual ACM Symposium on Principles of Programming Languages*, Paris, France, January 1997. [↑](#footnote-ref-2)
3. http://physics.nist.gov/cuu/pdf/sp811.pdf [↑](#footnote-ref-3)
4. Section 6.1.2 Plurals on page 12. [↑](#footnote-ref-4)
5. Section 9.2 Plurals on page 43. [↑](#footnote-ref-5)
6. While the parentheses make the associates more explicit, many would argue that they make it more difficult to actually read what has been written. Therefore, in most situations, only use parentheses when there is true ambiguity or it is critical to focus the reader’s attention. [↑](#footnote-ref-6)
7. Permission for use has been received from William Riddle. [↑](#footnote-ref-7)
8. The use of the word “canonical” in this context means “adhering to what is commonly accepted”. [↑](#footnote-ref-8)
9. Typically, food is sold by dry weight as opposed to volume, but let’s ignore that for the purpose of this example. [↑](#footnote-ref-9)